My researches

My reasearch focusses on:

- Deformation of polytopes:
  - ⋆ Deformation cone of nestohedra
  - ⋆ Deformation cone of graphical zonotopes
- Fiber polytopes:
  - ⋆ Monotone path polytopes of hypersimplices
  - ⋆ Fiber polytopes for $\text{Cyclic}(d, n) \to \text{Cyclic}(2, n)$
- Pivot polytopes:
  - ⋆ Pivot polytopes of cyclic polytopes
  - ⋆ Pivot polytopes of product of simplices

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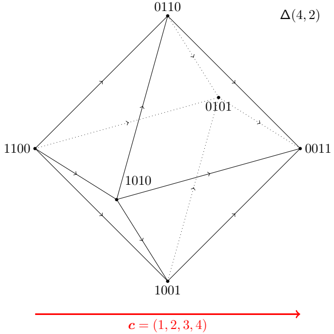
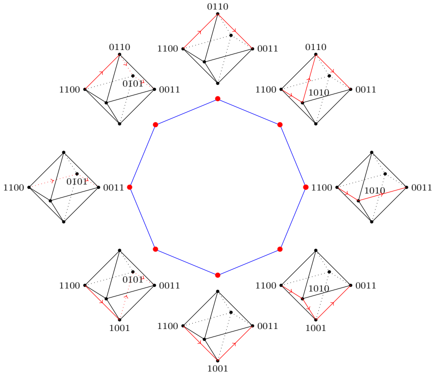
Monotone path polytopes of hypersimplex $(n, 2)$

Germain Poullot





1. Shadow vertex rule and monotone paths

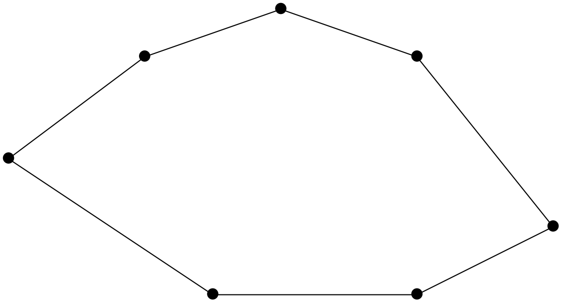
2. Hypersimplex $\Delta(n, k)$

3. Monotone path polytope of $\Delta(n, 2)$

Shadow vertex rule and monotone paths

Shadow vertex rule

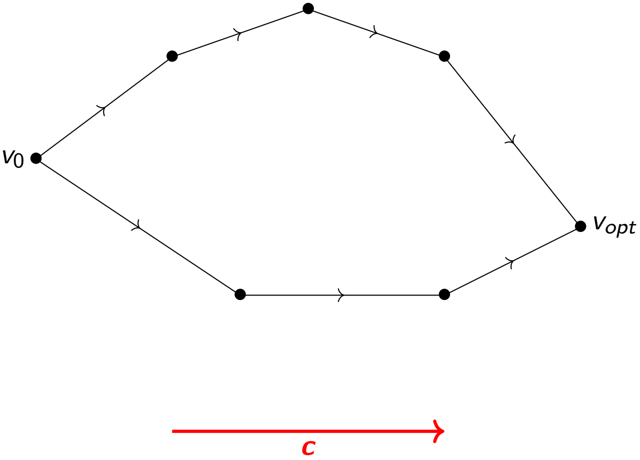
Optimization in dimension 2 (for linear programs):



Shadow vertex rule

Optimization in dimension 2 (for linear programs):

Goal: start at v_0 and find v_{opt}.



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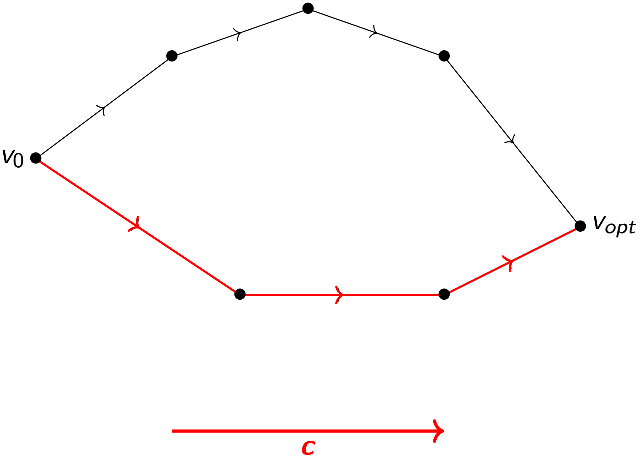

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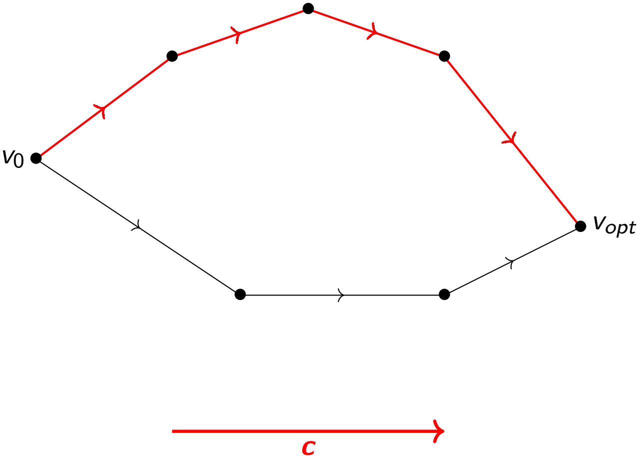
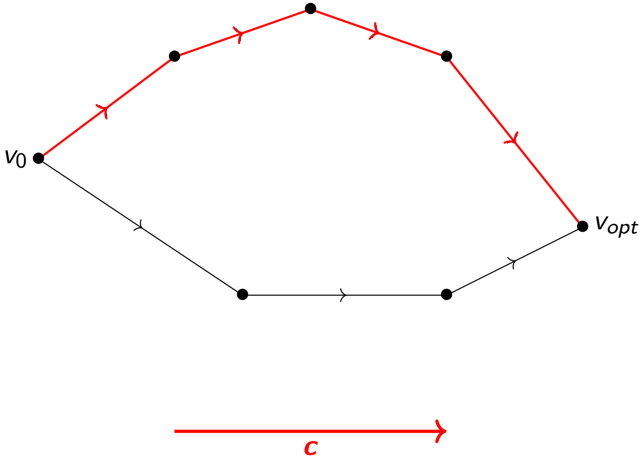


Shadow vertex rule

Optimization in dimension 2 (for linear programs): **EASY** !

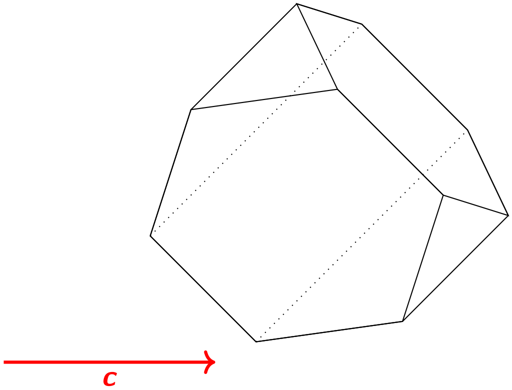
Goal: start at v_0 and find v_{opt}.



By convention, we always choose the upper path when optimizing.

Shadow vertex rule

Optimization in higher dimension: make it 2-dimensional !



Shadow vertex rule

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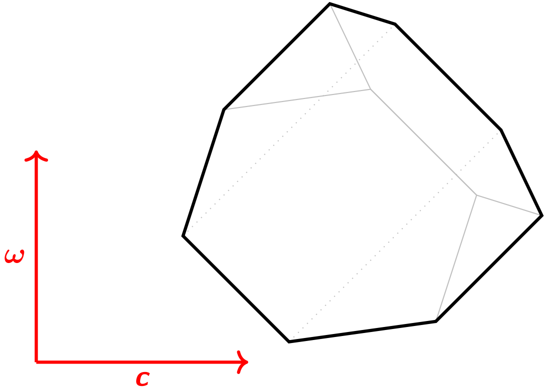

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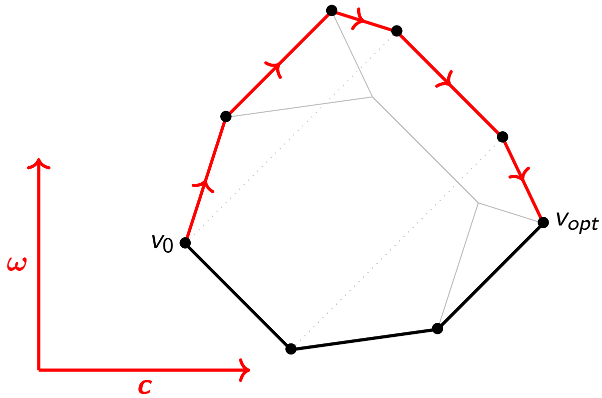

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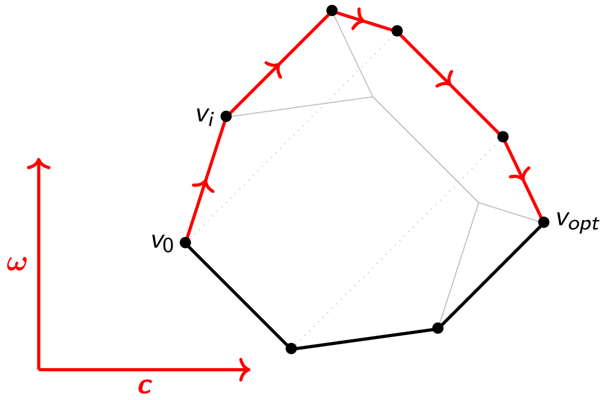
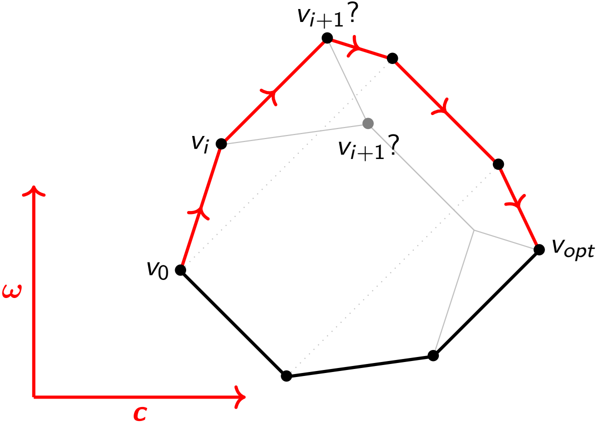


Optimization in higher dimension: make it 2-dimensional !



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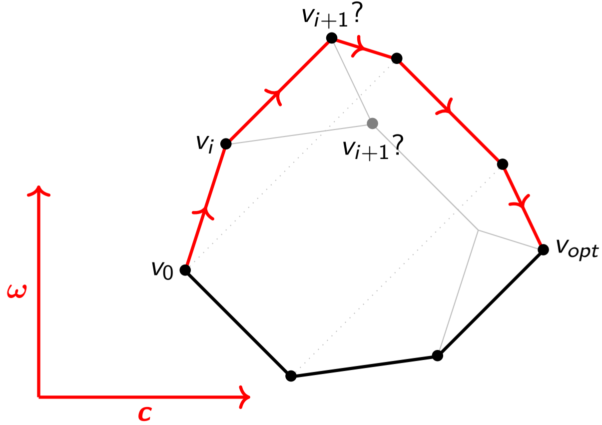

Shadow vertex rule (*i.e.* "take the neighbor with the best slope"):

$$A^{\omega}(v) = \text{argmax}\left\{\frac{\langle \boldsymbol{\omega}, u - v\rangle}{\langle \boldsymbol{c}, u - v\rangle}; u \text{ impr. neig. of } v\right\}$$

Monotone path polytope

Let $P \subset \mathbb{R}^d$ be a polytope.

Shadow vertex rule: $A^{\boldsymbol{\omega}}(v) = \text{argmax}\left\{\frac{\langle \boldsymbol{\omega}, u-v\rangle}{\langle \boldsymbol{c}, u-v\rangle}; u \text{ impr. neig. of } v\right\}$.

Coherent monotone path: A monotone path that can be obtained via the shadow vertex rule: $v_{i+1} = A^{\boldsymbol{\omega}}(v_i)$ for some $\boldsymbol{\omega}$.

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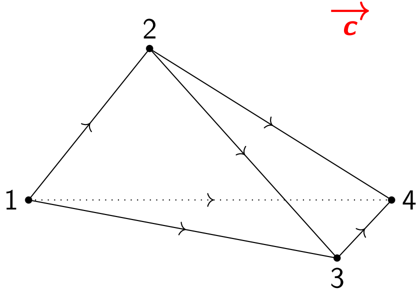
Monotone path polytope $\Sigma_\pi(\mathrm{P})$: Fiber polytope of $\mathrm{P} \xrightarrow{\pi} \mathrm{Q}$ with Q a segment. (Can be seen as a Minkowski sum of sections of P.)
The vertices of $\Sigma_\pi(\mathrm{P})$ are all coherent monotone paths.

Monotone path polytope of the simplex



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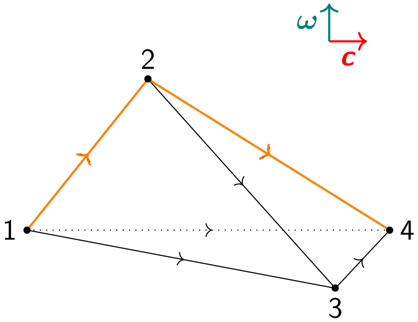

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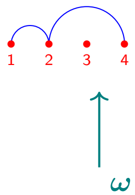

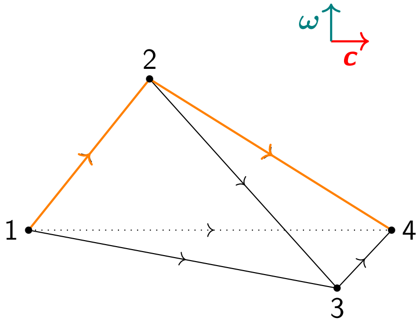

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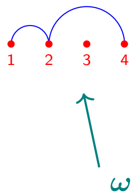

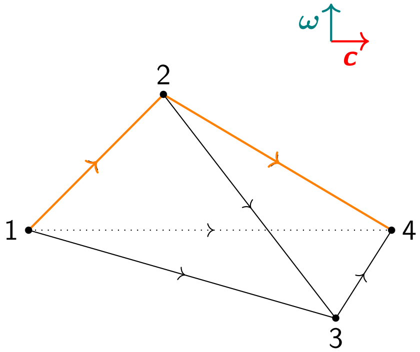

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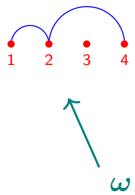

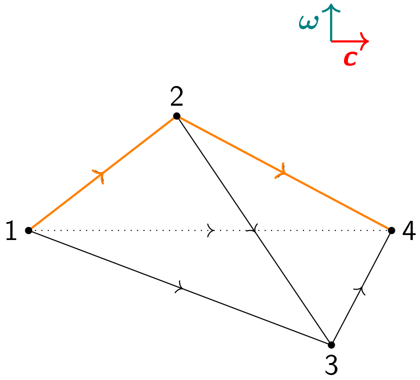

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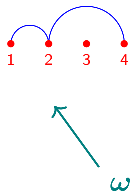

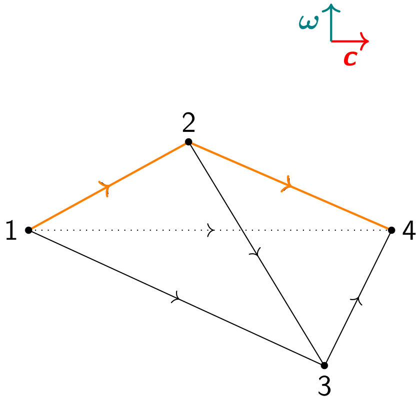

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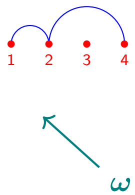

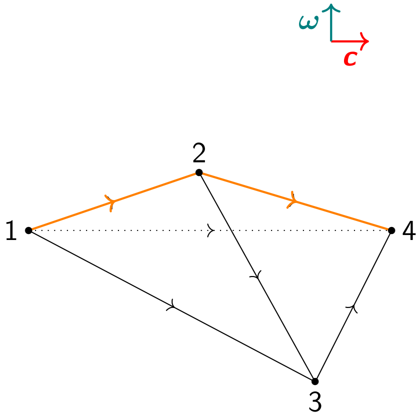

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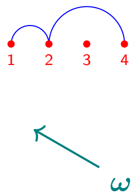

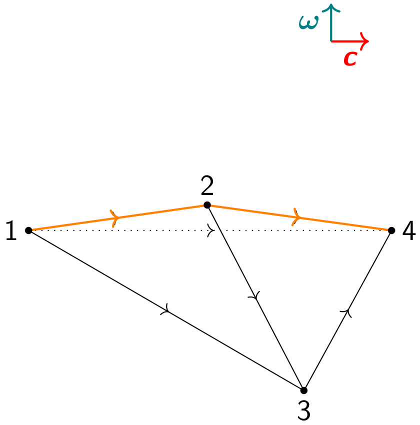

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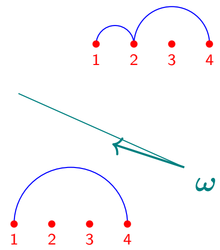

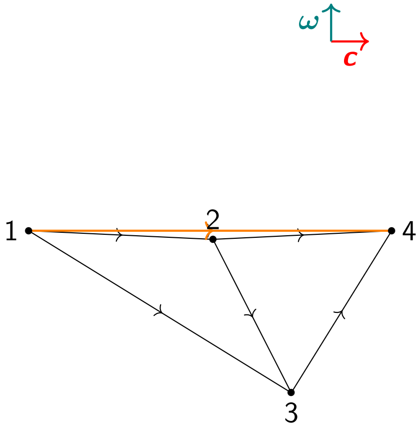

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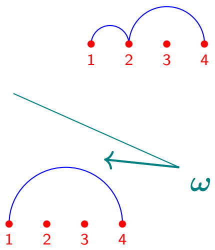

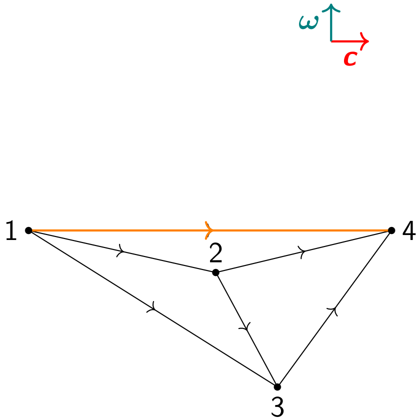

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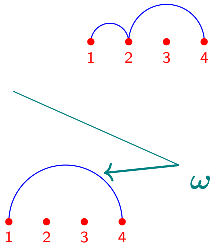

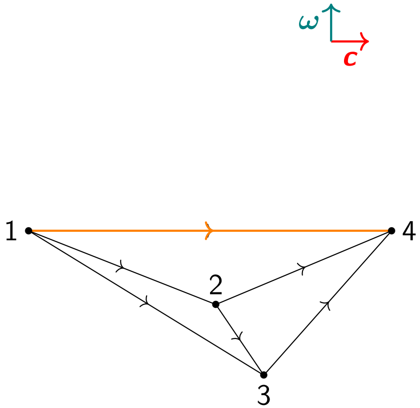

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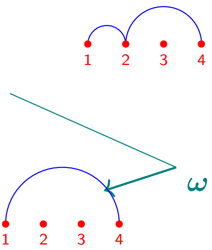

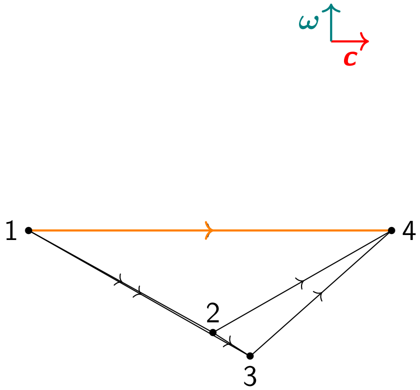

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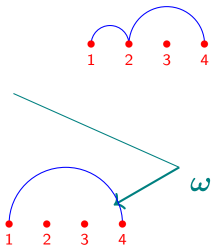

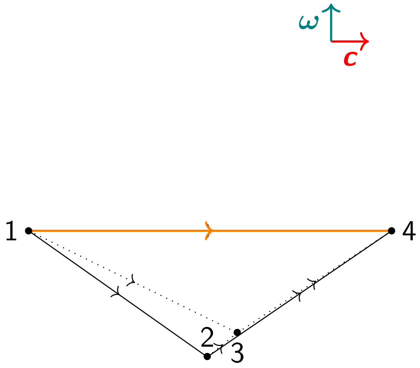

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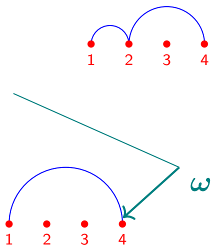

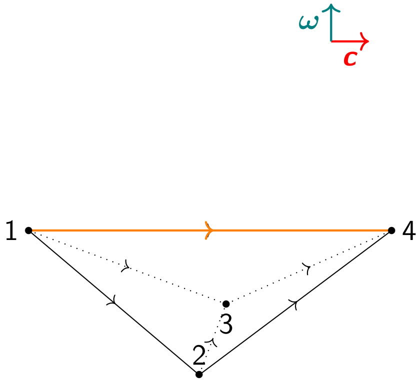

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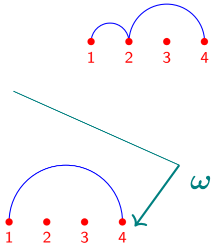

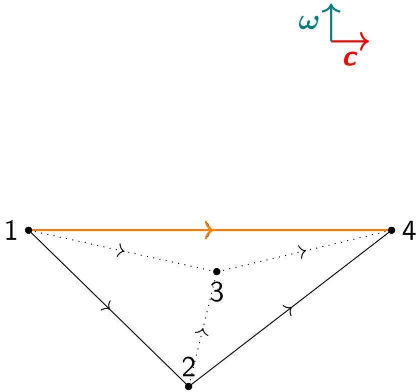

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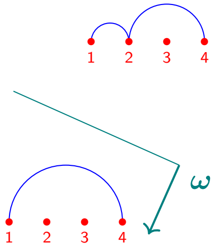

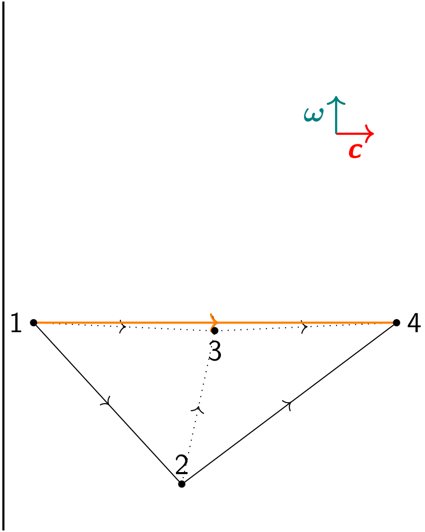

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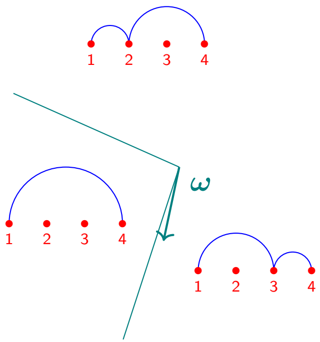

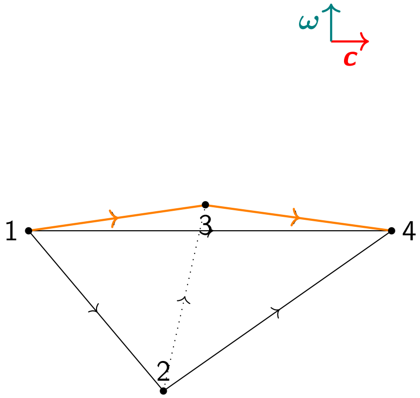

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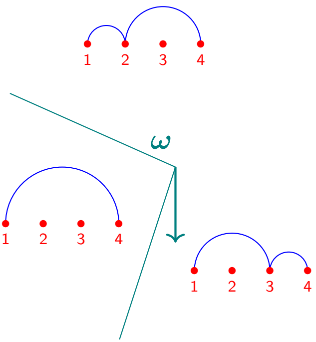

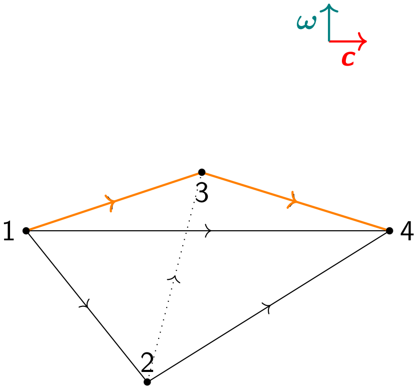

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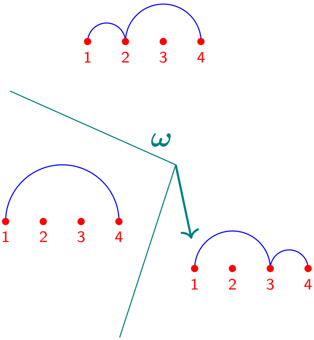

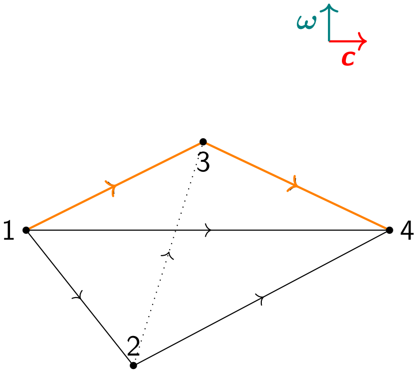

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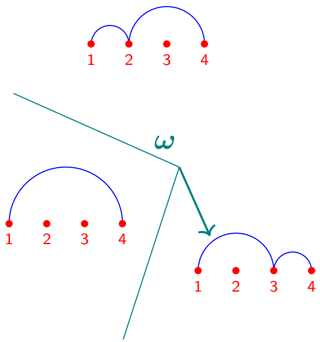

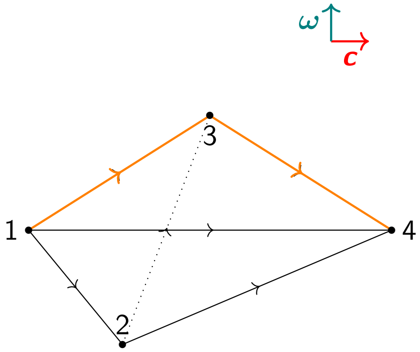

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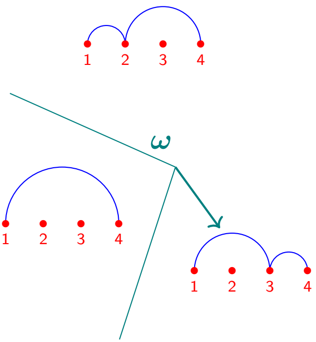

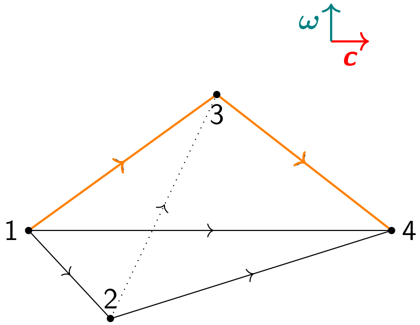

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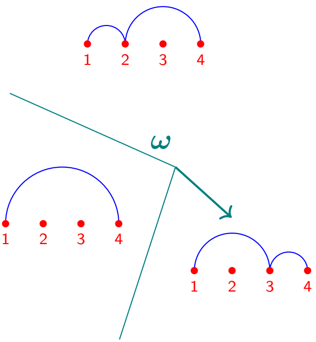

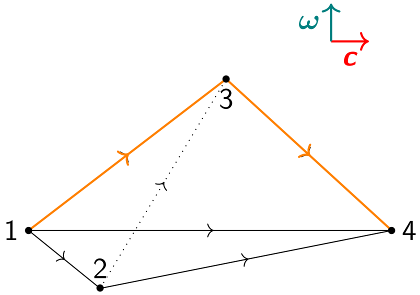

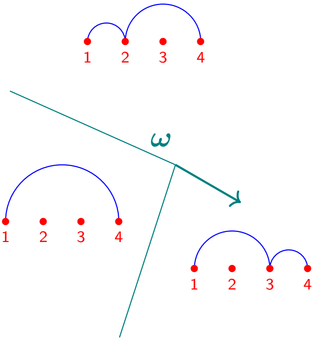
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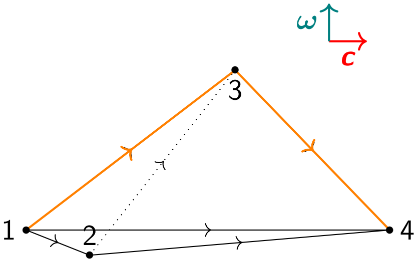
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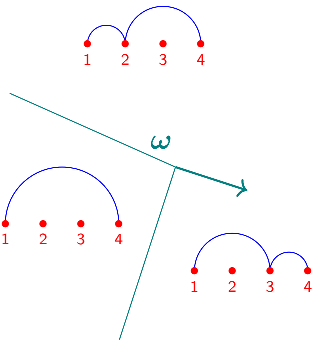
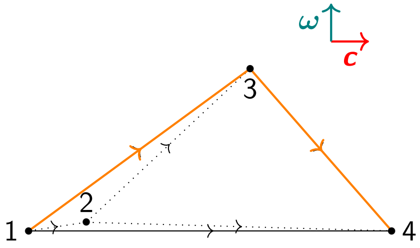
Monotone path polytope of the simplex





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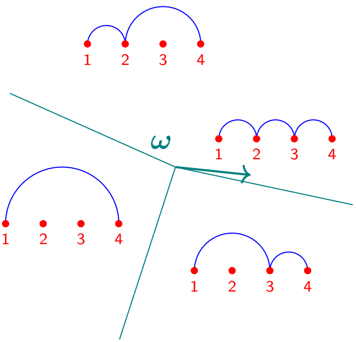

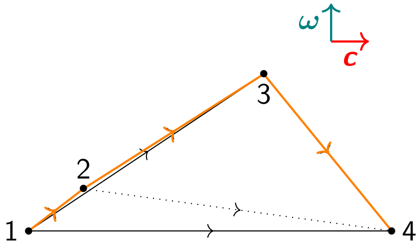

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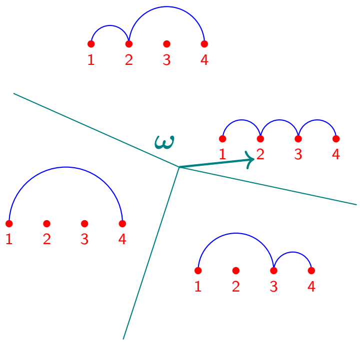

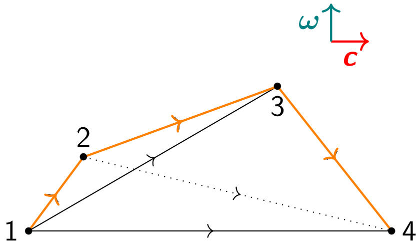

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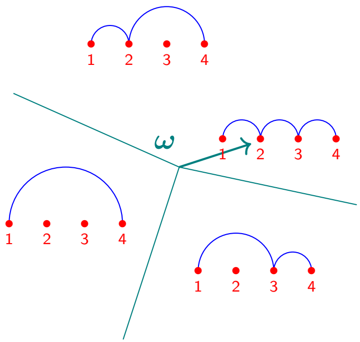

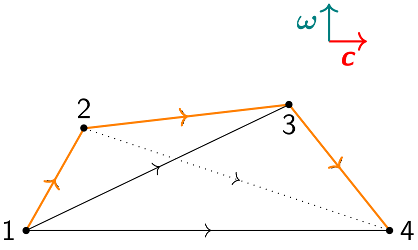

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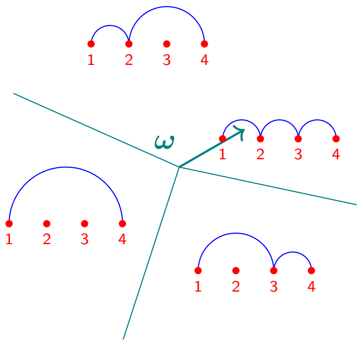

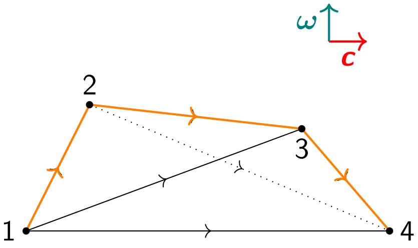

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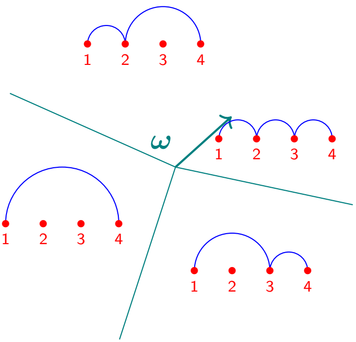

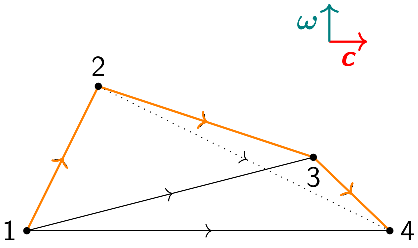

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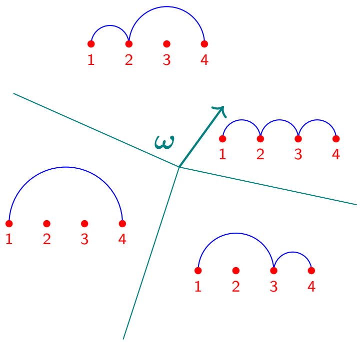

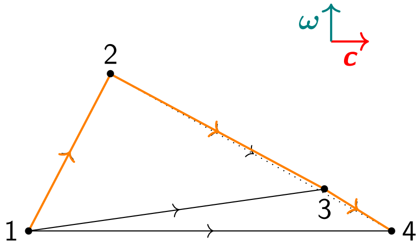

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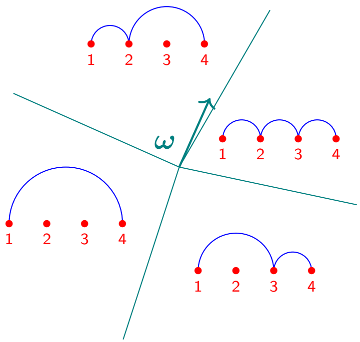

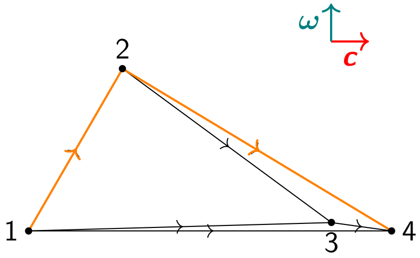

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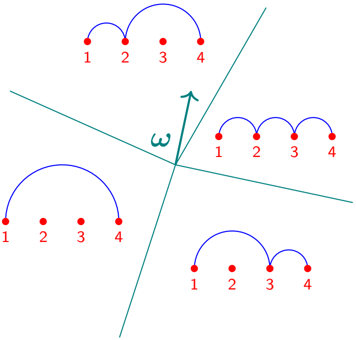

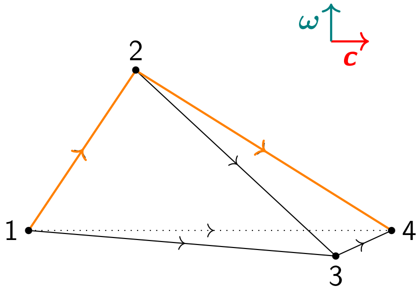

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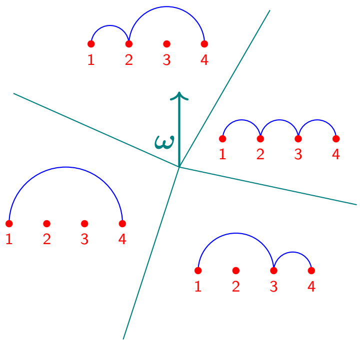
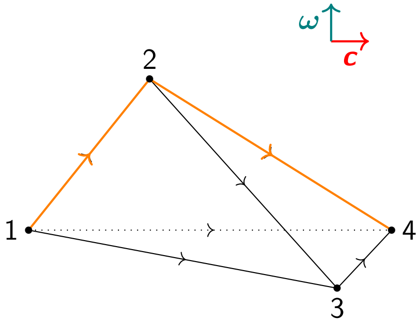
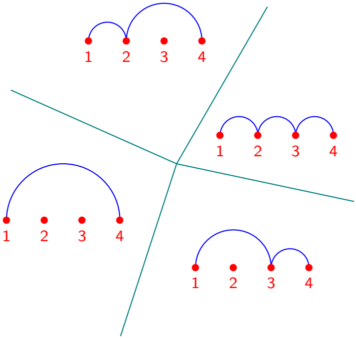
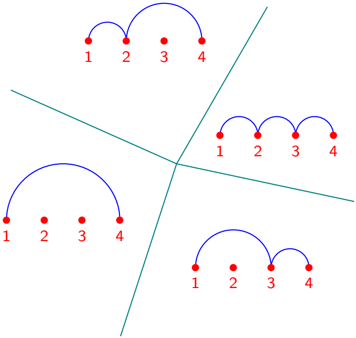






Monotone path fan $\pi_{\boldsymbol{c}}(\mathrm{P})$:
$\boldsymbol{\omega} \sim \boldsymbol{\omega}'$ iff they induce the same monotone path.
This gives a polytopal fan.



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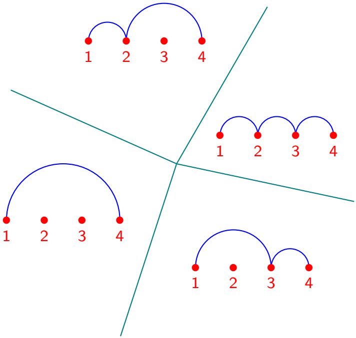
The monotone path polytope is the dual of this fan.

Case of the d-simplex



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For any d-simplex Δ_{d+1}, any π:

$$\Sigma_\pi(\Delta_{d+1}) = \mathrm{Cube}_{d-1}$$

For the cube Cube_d, any π:

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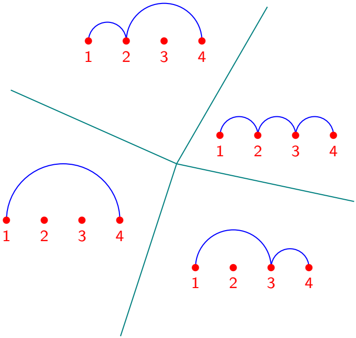

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$\Sigma_\pi(\Delta_{d+1})$:
A monotone path $= (v_0,$ part of the vertices, $v_{opt})$.
Choosing a monotone path $=$ Choosing a part of the $(d-1)$-remaining vertices.
Exercise: Prove all such paths are coherent.

Hypersimplex $\Delta(n, k)$

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Fix $n \geq 1$ and $k \in [1, n-1]$.

Definition

In $\mathbb{R}^n$, the *hypersimplex* $\Delta(n, k)$ is

$$\Delta(n, k) = \text{conv}\left\{\boldsymbol{v} \in \{0, 1\}^n \ : \ \sum v_i = k\right\}$$

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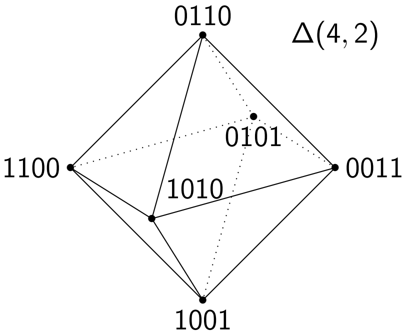
dimension: $n - 1$
Number of vertices: $\binom{n}{k}$

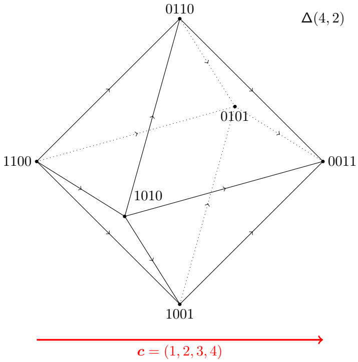
It is a section of the standard cube by an hyperplane.

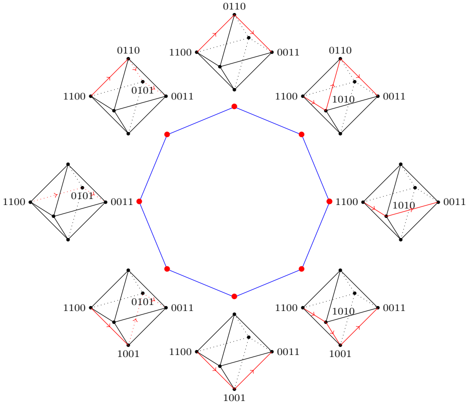
$\Delta(n, 1)$ and $\Delta(n, n-1)$ are simplices.

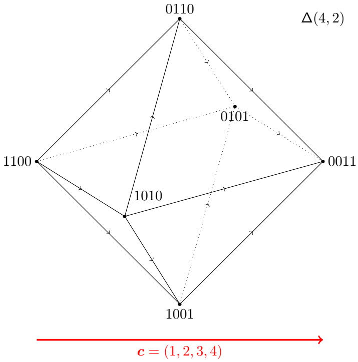


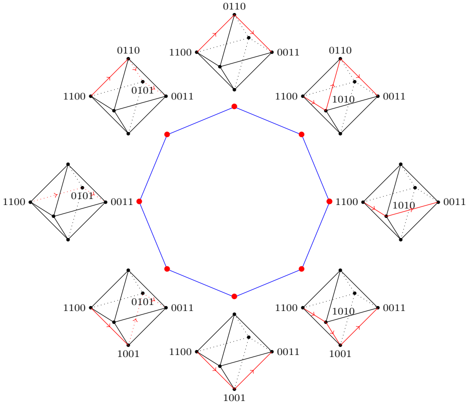
Monotone path polytope of $\Delta(n, 2)$

Monotone paths on $\Delta(n, k)$



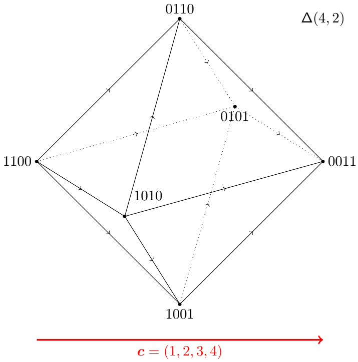


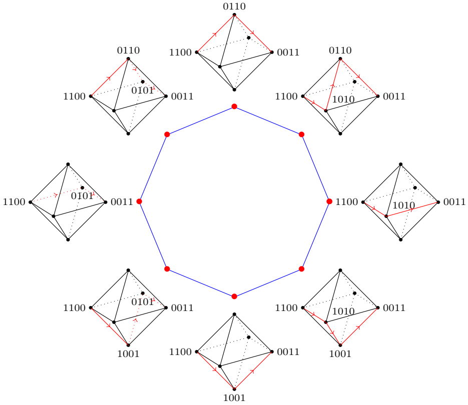




How many coherent monotone paths on $\Delta(n,2)$?

Monotone paths on $\Delta(n, k)$

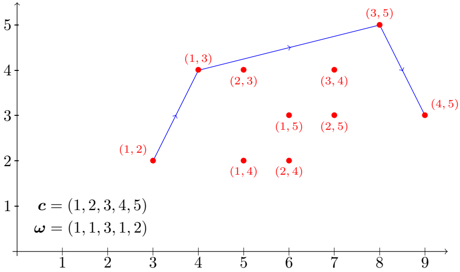


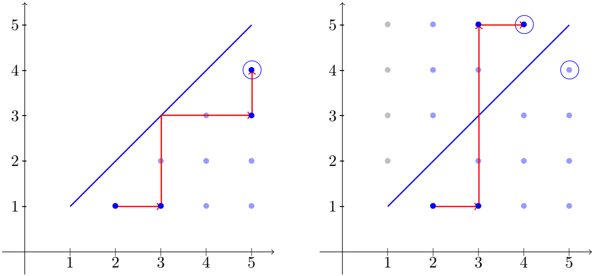


How many coherent monotone paths on $\Delta(n, 2)$?

n	4	5	6	7	8	...
Number of coherent paths	8	33	133	533	2133	???

Monotone paths on $\Delta(n, k)$





A coherent path with steps $2 \xrightarrow{1} 3$, $1 \xrightarrow{3} 5$, $3 \xrightarrow{5} 4$.

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Theorem (Necessary criterion)

When $i \xrightarrow{a} j$ preceeds $x \xrightarrow{z} y$ in the path, then if $x < j$, then $x = a$ or $j = z$.

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Theorem

For $k = 2$, this criterion is sufficient!

$\Rightarrow$ bijection between vertices of $\Sigma_\pi(\Delta(n, 2))$ and lattice paths with a simple property.

Induction process

We can inductively describe the lattice paths at stake.

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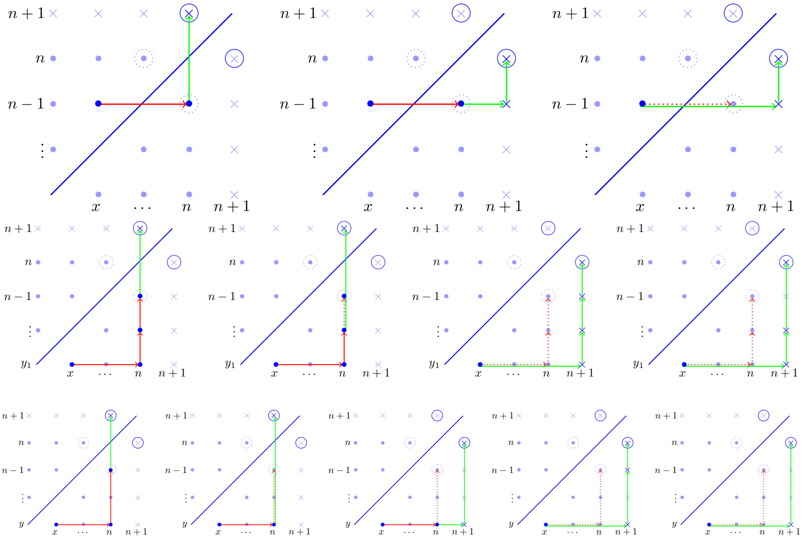

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Lemma

Number of coherent paths : $t_n + q_n + c_n$ *with*

$$\forall n \geq 4, \quad \begin{pmatrix} t_{n+1} \\ q_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} t_n \\ q_n \\ c_n \end{pmatrix} \quad \textit{with} \quad \begin{pmatrix} t_4 \\ q_4 \\ c_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

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Theorem (Number vertices of $\Sigma_\pi(\Delta(n,2))$)

For $n \geq 4$, there are $\frac{1}{3}(25 \times 4^{n-4} - 1)$ coherent paths of size n. This is the number of vertices of $\Sigma_\pi(\Delta(n,2))$.

n	4	5	6	...	n
Nb of coherent paths	8	33	133	...	$\frac{1}{3}(25 \times 4^{n-4} - 1)$

A conjecture on log-concavity

De Loera's conjecture

Conjecture (De Loera)

For any polytope P and generic objective function $\boldsymbol{c}$, the sequence $(N_\ell\ ;\ \ell \geq 1)$ of number of coherent monotone paths on P of length ℓ is a log-concave sequence.

Induction process (better)

$v_{n,\ell}$ = number of coherent paths on $\Delta(n,2)$ of length ℓ.
$V_n(z) = \sum_\ell v_{n,\ell} z^\ell$, generating polynomial

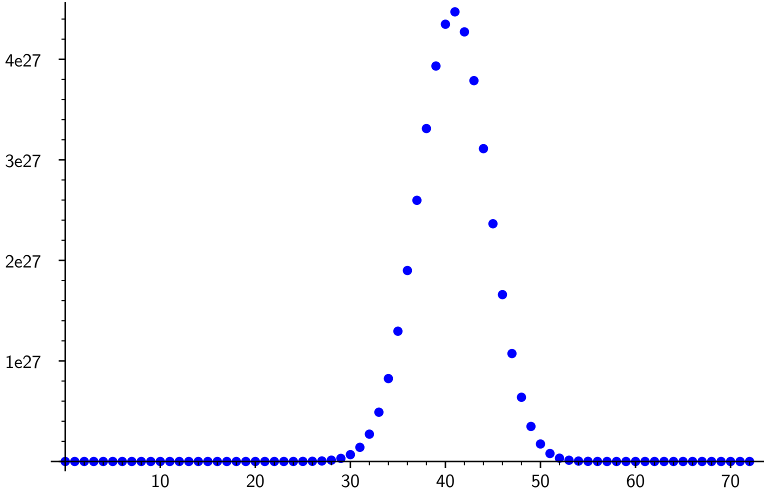
Lemma

Number of coherent paths : $V_n = T_n + Q_n + C_n$ *with*

$$\forall n \geq 4, \quad \begin{pmatrix} T_{n+1} \\ Q_{n+1} \\ C_{n+1} \end{pmatrix} = \begin{pmatrix} z & 1+z & 1+z \\ 0 & 1+z & z \\ z+z^2 & 0 & 1+z \end{pmatrix} \begin{pmatrix} T_n \\ Q_n \\ C_n \end{pmatrix}$$

$$\textit{with} \quad \begin{pmatrix} T_4 \\ Q_4 \\ C_4 \end{pmatrix} = \begin{pmatrix} z^4 + 2z^3 \\ z^4 \\ 2z^4 + 2z^3 \end{pmatrix}$$

Induction process (better)



$\left(v_{n,\ell}\right)_\ell$ seems to be log-concave (here for $n = 50$)...

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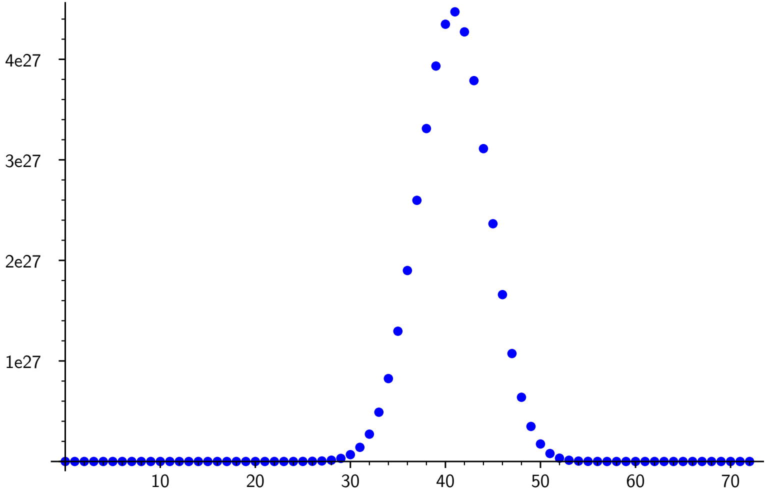

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but have resisted my attemps to prove it so.

Thank you for your attention!